Separation of the complex zeros of the Riemann zeta function

Herman te Riele, CWI Amsterdam

Symposium

Computing by the Numbers: Algorithms, Precision, and Complexity

in honor of the 60th birthday of Richard Brent

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Outline

- The Riemann hypothesis
- History
- Some “zeros” of $\zeta(s)$ with Mathematica
- From complex to real: the Rieman-Siegel function $Z(t)$
- Total number of zeros in a given part of the critical strip
- The Euler-Maclaurin formule voor $\zeta(s)$
- The Riemann-Siegel formula for $Z(t)$
- Gram points and Gram’s “Law”
- Rosser’s rule
- Separation of zeros of $Z(t)$ for $t \geq g_j$
- Stopping at g_l, knowing $\geq l - j$ zeros
- The sign of $Z(t)$

The Riemann hypothesis

The Riemann zeta function $\zeta(s)$ is the analytic function of $s = \sigma + it$ defined by:

$$\zeta(s) = \sum_{n=1}^{\infty} n^{-s}$$

for $\sigma > 1$, and by analytic continuation for $\sigma \leq 1, \sigma \neq 1$. Apart from “trivial” zeros at the negative even integers, all zeros of $\zeta(s)$ lie in the so-called critical strip $0 < \sigma < 1$.

The Riemann hypothesis is the conjecture that all nontrivial zeros

of $\zeta(s)$ lie on the so-called critical line $\sigma = 1/2$.

History

1903	15	Gram
1914	79	Backlund
1925	138	Hutchinson
1935	1,041	Titchmarsh
1953	1,104	Turing
1956	25,000	Lehmer
1958	35,337	Meller
1966	250,000	Lehman
1968	3,502,500	Rosser, Yohe, Schoenfeld
1979	81,000,001	Brent
1982	200,000,001	Brent, Van de Lune, Te Riele, Winter
1983	300,000,001	Van de Lune, Te Riele
1986	1,500,000,001	Van de Lune, Te Riele, Winter
2001	10,000,000,000	Van de Lune
2004	900,000,000,000	Wedeniwski
2004	10,000,000,000,000	Gourdon en Demichel

Some "zeros" of $\zeta(s)$ with Mathematica

```
In[39]:= FindRoot[Zeta[s] == 0, {s, 0}]
Out[39]= {s → -2.}
         {s → -1.9999999999999973`}

In[40]:= FindRoot[Zeta[s] == 0, {s, 0.4 + 12 I}]
Out[40]= {s → 0.5 + 14.1347 i}
         {s → 0.5000000000000071` + 14.134725141734693` i}
```

From complex to real: the Riemann-Siegel function $Z(t)$

$\zeta(s)$ satisfies a functional equation which may be written in the form

$$\xi(s) = \xi(1-s), \text{ with } \xi(s) = \pi^{-s/2}\Gamma(s/2)\zeta(s).$$

It follows that, if

$$\theta(t) = \arg\left(\pi^{-\frac{1}{2}it}\Gamma(\frac{1}{4}+\frac{1}{2}it)\right) = \Im[\log\Gamma(\frac{1}{4}+\frac{1}{2}it)] - \frac{t}{2}\log\pi,$$

then

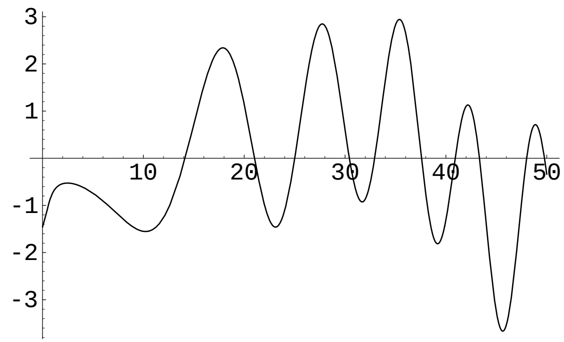
$$Z(t) = e^{i\theta(t)}\zeta(\frac{1}{2}+it)$$

is real (for real t). Since $|Z(t)| = |\zeta(\frac{1}{2}+it)|$, the zeros of Z are the imaginary parts of the zeros of $\zeta(s)$ on the critical line.

$Z(t)$ is known as the Riemann-Siegel function.

Some plots of $Z(t)$ with Mathematica

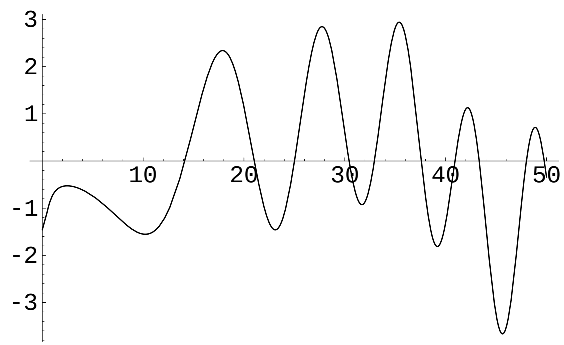
```
In[63]:= Plot[RiemannSiegelZ[t], {t, 0, 50}]
```




```
Out[63]= - Graphics -
```

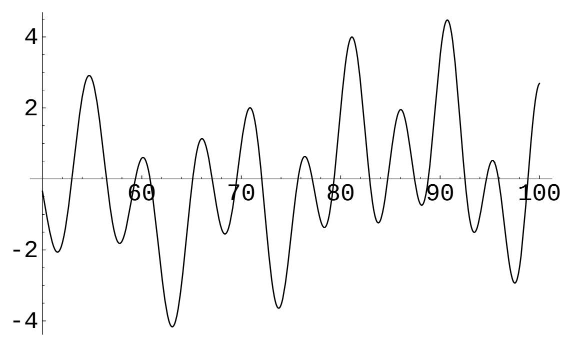
Some plots of $Z(t)$ with Mathematica

```
In[63]:= Plot[RiemannSiegelZ[t], {t, 0, 50}]
```




```
Out[63]= - Graphics -
```

```
In[64]:= Plot[RiemannSiegelZ[t], {t, 50, 100}]
```

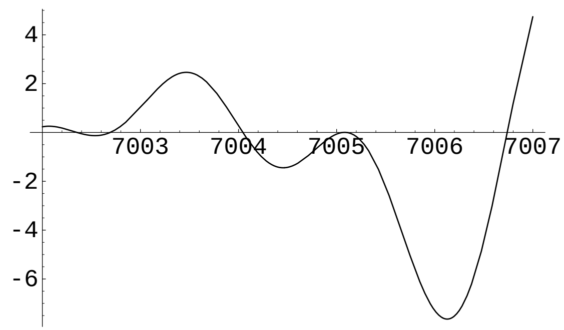



```
Out[64]= - Graphics -
```

Another plot of $Z(t)$

Close zeros: Lehmer phenomenon

```
In[61]:= Plot[RiemannSiegelZ[t], {t, 7002, 7007}]
```

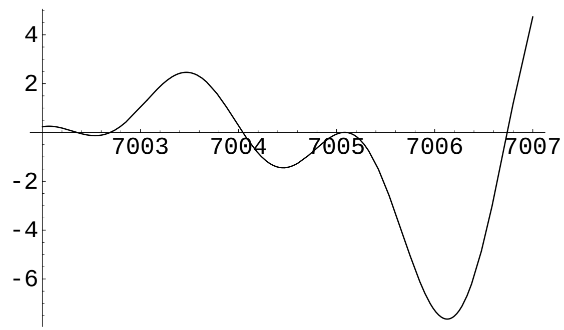



```
Out[61]= - Graphics -
```

Another plot of $Z(t)$

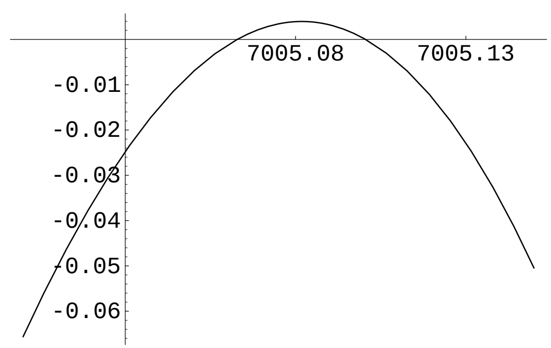
Close zeros: Lehmer phenomenon

```
In[61]:= Plot[RiemannSiegelZ[t], {t, 7002, 7007}]
```




```
Out[61]= - Graphics -
```

```
In[62]:= Plot[RiemannSiegelZ[t], {t, 7005, 7005.15},
          Ticks → {{7005.03, 7005.08, 7005.13}, Automatic}]
```

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Out[62]= - Graphics -
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Total number of zeros in a given part of the critical strip

Backlund proved around 1912 that the total number $N(T)$ of complex zeros ρ of $\zeta(s)$ with $0 < \Im(\rho) < T$ (counting multiplicities) can be expressed as

$$N(T) = \frac{\theta(T)}{\pi} + 1 + \frac{1}{\pi}\Im\left\{\int_C \frac{\zeta'(s)}{\zeta(s)}ds\right\},$$

where C is the broken line segment from $1+\epsilon$ (for some $\epsilon > 0$) to $1+\epsilon+iT$ to $\frac{1}{2}+iT$.

Backlund observed that if $\Re\zeta \neq 0$ on C, then this formula suffices to determine $N(T)$ as the nearest integer to $\theta(T)/\pi + 1$.

Expansion:

$$\theta(t) = \frac{t}{2}\log\frac{t}{2\pi} - \frac{t}{2} - \frac{\pi}{8} + \frac{1}{48t} + \mathcal{O}(t^{-3}), \text{ as } t \to \infty.$$

Example: $\theta(100)/\pi + 1 = 29.0024\ldots$.

The Euler-Maclaurin formule for $\zeta(s)$

$$\zeta(s) = \sum_{j=1}^{n-1} j^{-s} + \frac{1}{2}n^{-s} + \frac{n^{1-s}}{s-1} + \sum_{k=1}^{m} T_{k,n}(s) + E_{m,n}(s),$$

where

$T_{k,n} = \frac{B_{2k}}{(2k)!} n^{1-s-2k} \prod_{j=0}^{2k-2}(s+j)$, $|E_{m,n}| < \left| T_{m+1,n}\left(\frac{s+2m+1}{\sigma+2m+1}\right)\right|$

for all $m \geq 0$, $n \geq 1$ en $\sigma = \Re(s) > -(2m+1)$.

$B_{2k}, k = 1, 2, \ldots$ are the Bernoulli-numbers:

$B_2 = 1/6, B_4 = -1/30, B_6 = 1/42, \ldots$.

For $s = \frac{1}{2} + it$ a good choice for m, n is: $m \approx \sqrt{n}$ and $n \approx t/(2\pi)$.

Complexity: $\mathcal{O}(t)$.

The Riemann-Siegel formula for $Z(t)$

Let $\tau := t/(2\pi)$, $m := \lfloor \tau^{1/2} \rfloor$, and $z := 2(\tau^{1/2} - m) - 1$, then

$$Z(t) = 2\cos(\theta(t)) + 2\sum_{k=2}^{m} k^{-1/2}\cos[t\log k - \theta(t)] +$$

$$+(-1)^{m+1}\tau^{-1/4}\sum_{i=0}^{n}\Phi_i(z)(-1)^i\tau^{-i/2} + R_n(\tau),$$

with $\Phi(z) = \ldots$ and $R_n(\tau) = \mathcal{O}(\tau^{-(2n+3)/4}), n \geq -1, \tau > 0.$
For $\tau > 32$ ($t > 200$) we have $|R_n(\tau)| < d_n\tau^{-(2n+3)/4}$ with
$d_0 = 0.032$, $d_1 = 0.0054$, $d_2 = 0.00045$ and $d_3 = 0.0005$

(Doctor's Thesis Gabcke). Complexity: $\mathcal{O}(\sqrt{t})$.

Gram points and Gram's "Law"

The Gram point $g_m, m = -1, 0, 1, \ldots,$ is the unique solution in $[7, \infty)$ of the equation

$$\theta(x) = m\pi.$$

The value of the first term $2\cos(\theta(t))$ in the Riemann-Siegel formula in the Gram point $t = g_m$ equals

$$2\cos(\theta(g_m)) = 2\cos(m\pi) = 2(-1)^m.$$

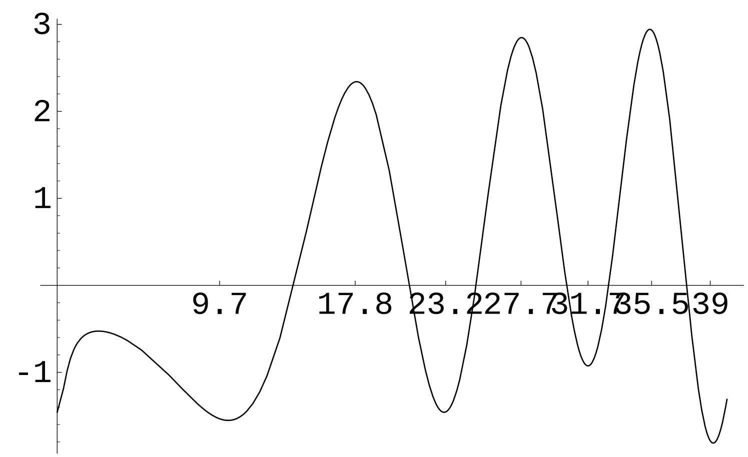
Gram's "Law" is the tendency of the zeros of $Z(t)$ to alternate with the Gram points (based on the expectation that the first term in the RS formula dominates).

Unfortunately, this "Law" fails infinitely often, but it is known that on average there is precisely one zero of $Z(t)$ inbetween two consecutive Gram points.

Illustration of Gram's "Law"

$g_{-1}, g_0, g_1, g_2, g_3, g_4, g_5 =$
$9.7, 17.8, 23.2, 37.7, 31.7, 35.5, 39.0$

```
In[88]:= Plot[RiemannSiegelZ[t], {t, 0, 40},
          Ticks → {{9.7, 17.8, 23.2, 27.7, 31.7, 35.5, 39.0, 42.34}, Automatic}]
```



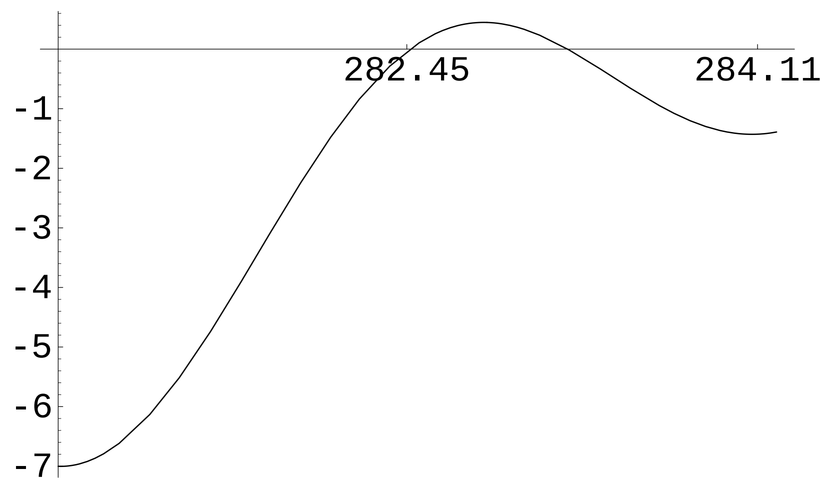

```
Out[88]= - Graphics -
```

First violation of Gram's "Law"

$g_{125}, g_{126}, g_{127} =$
$280.80, 282.45, 284.1$

In[112]:=

```
Plot[RiemannSiegelZ[t], {t, 280.8, 284.2}, Ticks → {{280.8, 282.45, 284.11}, Automatic}]
```



Out[112]=

- Graphics -

Rosser's Rule

A Gram point g_n is called good if $(-1)^n Z(g_n) > 0$, and bad otherwise.
A Gram block is an interval $[g_n, g_{n+k})$, $k \geq 1$, such that both g_n and g_{n+k} are good Gram points whereas the intermediate points $g_{n+1}, \ldots, g_{n+k-1}$ are bad points.
Rosser's Rule: Gram blocks of length k contain exactly k zeros.

Richard Brent found the first exception to Rosser's Rule, namely the Gram block of length 2: $[g_n, g_{n+2})$ with $n = 13999525$. This contains no zeros but it is followed by the Gram block $[g_{n+2}, g_{n+3})$ with three zeros.

Example: Sign pattern of $Z(t)$ on Gram block of length 8

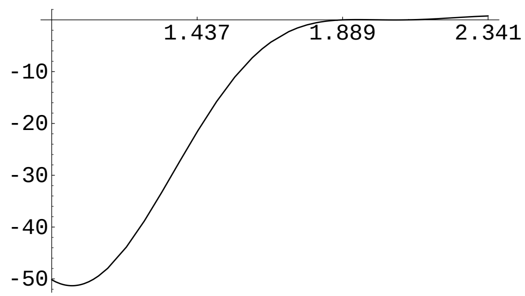
Let e be an even (w.l.o.g.) positive integer.

k	e	$e+1$	$e+2$	$e+3$	$e+4$	$e+5$	$e+6$	$e+7$	$e+8$
$(-1)^k Z(g_k)$	$+$	$-$	$-$	$-$	$-$	$-$	$-$	$-$	$+$
$(-1)^k$	$+$	$-$	$+$	$-$	$+$	$-$	$+$	$-$	$+$
$Z(g_k)$	$+$	$+$	$-$	$+$	$-$	$+$	$-$	$+$	$+$

First violation of Rosser's Rule

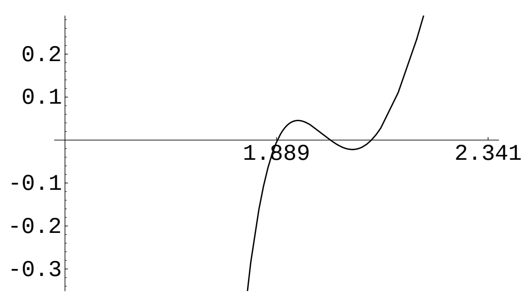
$$Z(g_{13999525}, g_{13999526}, g_{13999527}, g_{13999528}) =$$
$$Z(6820050.985, 6820051.437, 6820051.889, 6820052.341) =$$
$$(-50.2, -21.62, -0.0045, +0.752)$$

```
In[8]:= Plot[RiemannSiegelZ[t + 6820050], {t, 0.984896665, 2.341223716},
         Ticks → {{0.985, 1.437, 1.889, 2.341}, Automatic}]
```




```
Out[8]= - Graphics -
```

```
In[10]:= Plot[RiemannSiegelZ[t + 6820050], {t, 1.437005684, 2.341223716},
          Ticks → {{1.437, 1.889, 2.341}, Automatic}]
```

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Out[10]= - Graphics -
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Separation of zeros of $Z(t)$ for $t \geq g_j$

Violations of Rosser's Rule are extremely rare, so Rosser's Rule is a very useful heuristic to verify the Riemann hypothesis:

suppose that $j+1$ sign changes of $Z(t)$ have been found in (g_{-1}, g_j), then evaluate $Z(g_{j+1}), Z(g_{j+2}), \ldots$ until the next good Gram point g_{j+k} (following g_j) is found. In roughly 70% of the cases this is g_{j+1}, i.e., $k=1$. $Z(g_j)$ and $Z(g_{j+1})$ then have different signs, so we have proved the existence of at least one zero between g_j and g_{j+1}. The process is continued then with g_j replaced by g_{j+1}.

Separation, the "missing two" zeros

If $k \geq 2$ then we have $k - 2$ sign changes on $[g_j, g_{j+k})$ and we try to find the "missing two" zeros on this Gram block.
If we succeed, then j is replaced by $j + k$ and the process is continued; if after a large number of Z-evaluations our program does not succeed, then either:
1 the block $[g_j, g_{j+k})$ does contain a pair of (not-yet-found) very close zeros –and the precision of the computations must be increased to find them–, or
2 the "missing two" zeros are found in the preceding or following Gram block (this occurs only roughly 30 times in every million consecutive Gram points), or
3 ???!!!.

A plot of $Z(t)$ on $[g_j, g_{j+k})$ and neighbouring Gram blocks usually

reveals where the "missing two" zeros are to be found.

Stopping at g_l, knowing $\geq l+1$ zeros

Theorem (Littlewood, Turing, Lehman, Brent)
If K consecutive Gram blocks with union $[g_l, g_p)$ satisfy Rosser's Rule, where

$$K \geq 0.0061 \log^2(g_p) + 0.08 \log(g_p),$$

then

$$N(g_l) \leq l+1 \text{ and } N(g_p) \geq p+1.$$

Gourdon, verifying RH until zero # 10^{13}, has $l = 10^{13} - 1$ with $g_l = 2,445,999,556,030.34\ldots$, $\log(g_l) = 28.52\ldots$, and

$$0.0061 \log^2(g_l) + 0.08 \log(g_l) \approx 7.25,$$

so $K = 8$ was sufficient for proving that $N(g_{10^{13}-1}) = 10^{13}$.

The sign of $Z(t)$

Applying backward error analysis to the evaluation of the Riemann-Siegel sum, Brent gave the following rigorous bound for the error in the computed value $\tilde{Z}(t)$ of $Z(t)$ with a fast, single-precision method:

$$|\tilde{Z}(t) - Z(t)| \leq (2 \times 10^{-5} + 5 \times 10^{-16}\tau \log(\tau) + 3\tau^{-2})\tau^{1/4},$$

where $\tau = t/(2\pi)$.

For the range for which Brent verified RH (75,000,000 zeros), the right-hand side is < 0.001. Brent gave a similar, but much more accurate, bound for a slower, double-precision method which was invoked when the fast method could not determine the sign of $Z(t)$ with certainty.

Conclusion

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Thanks for your cooperation, Richard, and:

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Thanks for your cooperation, Richard, and:

Happy Birthday!